TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 356, Number 5, Pages 2001–2023
S 0002-9947(03)03377-4
Article electronically published on November 12, 2003

VARYING THE TIME-FREQUENCY LATTICE OF GABOR FRAMES

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Abstract. A Gabor or Weyl-Heisenberg frame for $L^2(\mathbb{R}^d)$ is generated by time-frequency shifts of a square-integrable function, the Gabor atom, along a time-frequency lattice. The dual frame is again a Gabor frame, generated by the dual atom. In general, Gabor frames are not stable under a perturbation of the lattice constants; that is, even for arbitrarily small changes of the parameters the frame property can be lost.

In contrast, as a main result we show that this kind of stability does hold for Gabor frames generated by a Gabor atom from the modulation space $M^1(\mathbb{R}^d)$, which is a dense subspace of $L^2(\mathbb{R}^d)$. Moreover, in this case the dual atom depends continuously on the lattice constants. In fact, we prove these results for more general weighted modulation spaces. As a consequence, we obtain for Gabor atoms from the Schwartz class that the continuous dependence of the dual atom holds even in the Schwartz topology. Also, we complement these main results by corresponding statements for Gabor Riesz sequences and their biorthogonal system.

1. Introduction and main results

We will discuss in general terms the stability of Gabor frames for $L^2(\mathbb{R}^d)$ under small perturbations of the lattice constants, given the assumption that the Gabor atom belongs to a suitable modulation space. The standard case is the use of time-frequency lattices of the form $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$, with $a, b > 0$, and we will use this setting to formulate the questions which motivated our work. Yet our approach is not restricted to this case. We work in the context of general time-frequency lattices of the form $\Lambda = L\mathbb{Z}^{2d}$, where L is in $GL(\mathbb{R}^{2d})$, that is, L is an invertible matrix of size $2d \times 2d$. Thus, the matrix L represents the lattice parameters for a time-frequency lattice in the most general way. The standard case is obtained by chosing $L = \left(\begin{smallmatrix} aI & 0 \\ 0 & bI \end{smallmatrix}\right)$, where I denotes the $d \times d$ identity matrix.

We introduce the basic notations in Gabor analysis in a form which is suitable for this general framework.

Given $\lambda \in \mathbb{R}^{2d}$, the time-frequency shift $\pi(\lambda)f$ of a function $f \in L^2(\mathbb{R}^d)$ is defined by

$$\pi(\lambda)f(t) = e^{2\pi i\omega t} f(t-x), \qquad \lambda = (x,\omega) \in \mathbb{R}^{2d}. \tag{1.1}$$

Received by the editors April 29, 2002 and, in revised form, April 9, 2003.
2000 *Mathematics Subject Classification.* Primary 42C15; Secondary 47B38, 81R30, 94A12.
Key words and phrases. Gabor frame, Weyl-Heisenberg frame, dual atom, Riesz basis, stability, perturbation, time-frequency lattice, modulation space, twisted convolution, coherent states.
The second author was supported by the Austrian Science Fund FWF, grants P-14485 and J-2205.

A function $g \in L^2(\mathbb{R}^d)$ and an invertible matrix $L \in GL(\mathbb{R}^{2d})$ generate a Gabor system $\{\pi(Lk)g\}_{k\in\mathbb{Z}^{2d}} \subseteq L^2(\mathbb{R}^d)$, which consists of the time-frequency shifts of the Gabor atom g along the time-frequency lattice $\Lambda = L\mathbb{Z}^{2d}$. If a Gabor system is a frame for $L^2(\mathbb{R}^d)$, then it is called a Gabor frame. This is the case if the frame operator

$$Sf = \sum_{k\in\mathbb{Z}^{2d}} \langle f, \pi(Lk)g\rangle \pi(Lk)g, \qquad f \in L^2(\mathbb{R}^d), \tag{1.2}$$

is bounded and invertible on $L^2(\mathbb{R})$. In this case, $\widetilde{g} = S^{-1}g \in L^2(\mathbb{R}^d)$ is called the canonical dual atom, and the frame identity

$$f = \sum_{k\in\mathbb{Z}^{2d}} \langle f, \pi(Lk)\widetilde{g}\rangle \pi(Lk)g = \sum_{k\in\mathbb{Z}^{2d}} \langle f, \pi(Lk)g\rangle \pi(Lk)\widetilde{g} \tag{1.3}$$

holds for all $f \in L^2(\mathbb{R}^d)$.

To motivate our results, consider the standard case of dimension $d = 1$ with time-frequency lattice $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, where $a, b > 0$. A fundamental question in Gabor analysis is to characterize those triples (g, a, b) which generate a Gabor frame. For too sparse lattices ($ab > 1$) no frames exist. For the standard von Neumann lattice ($a = b = 1$) frames, even orthonormal bases exist; however, they are not well time-frequency localized since according to the Balian-Low theorem a good time-frequency concentration of both the atom and its dual is impossible at critical density ($ab = 1$), see [10, Sec. 4.1], [25, Sec. 8.4]. For oversampled time-frequency lattices ($ab < 1$), Gabor frames exist, including examples with excellent time-frequency localization.

In this paper, we address the problem of varying the time-frequency lattice constants (a, b) for Gabor frames, mainly driven by the following questions.

Question I. Let $g = 1_{[0,1]}$, the characteristic function of an interval. The "good" lattice constants (a, b) such that (g, a, b) generates a Gabor frame form a surprisingly strange subset of $\mathbb{R}_+ \times \mathbb{R}_+$, known as Janssen's tie [33], see also Example 5.3. This set is not an open set. For many pairs (a, b) which generate a frame there exist other pairs (a', b') arbitrarily close to (a, b) which do not generate a frame. That is, even for an arbitrary small perturbation of (a, b) the frame property can be lost. Hence a Gabor frame generated by the characteristic function of an interval is, in general, not stable under a perturbation of the lattice constants. On the other hand, the Gaussian function does enjoy this kind of stability, see Question II. Thus, we ask: *Are there natural conditions on the Gabor atom which guarantee stability of the frame condition under the perturbation of the lattice parameters (a, b)?*

Question II. The Gaussian function $g(t) = e^{-\pi t^2}$ yields a Gabor frame for any lattice with $ab < 1$. In particular, the set of pairs (a, b) satisfying this condition is an open set for this choice of g. In other words, Gabor frames with this generator g are stable under (sufficiently small) perturbations of the lattice constants. In fact, it is known that the dual atom is again a Schwartz function for each of the perturbed Gabor frames. However, before the present paper it was not known in which way these dual atoms are related to each other. This leads to our second question: *If the stability of a Gabor frame under a perturbation of the lattice is indeed guaranteed, then do the dual atoms depend continuously on the lattice parameters?*

There is extensive literature concerning local perturbations of Gabor frames [1, 4, 5, 6, 8, 9, 12, 14, 24, 37, 38, 41]. However, the perturbation of the lattice constants

(a, b) is of a different type since it generates an arbitrarily large (i.e., non-local) perturbation of the individual lattice points (ak, bl) for large $k, l \in \mathbb{Z}$. The proofs for the local perturbation results are based on a Paley-Wiener type perturbation of the Gabor frame operator $S = S_{g,a,b}$. By nature, this approach cannot be adapted to our problem since, as a matter of fact, S does *not* depend continuously in the L^2-operator norm on the lattice constants (a, b), even for Schwartz atoms g. The two positive results concerning the perturbation of (a, b) obtained before this paper ([21, Sec. 3.6.3], [3, Thm. 3.5]) cover situations only where the lattice parameters are sufficiently small, and the Neumann series representing the inverse frame operator is absolutely convergent. Note that for general L^2-atoms, the frame condition may depend critically on (a, b) even for arbitrarily small $a, b > 0$ [16].

Besides the fact mentioned above that the frame operator depends in a non-continuous way on the lattice constants, there is another obstacle which the present work had to overcome. Previously, irrational time-frequency lattices ($ab \notin \mathbb{Q}$) have typically required very different techniques compared to rational latices. For example, compare [25, Sec. 13.2] with [25, Sec. 13.5] and [29]; see also [2, 28] and [25, p. 139, Remark 2]. A structural difference between rational and irrational time-frequency lattices is also apparent in the shape of Janssen's tie. However, for the present question of stability one cannot treat the rational and irrational cases separately. Any small perturbation of the lattice parameters involves rational and irrational lattices at the same time.

The present approach is a new strategy, based on the Janssen representation; the use of the appropriate class of windows; a continuity result for the twisted convolution; and the occasional use of Lebesgue dominated convergence. The motivating questions above are answered affirmatively in a general higher-dimensional arbitrary lattice setting. We will describe the main results now.

For a subspace $X \subseteq L^2(\mathbb{R}^d)$, define the set

$$F_X = \big\{(g, L) \in X \times GL(\mathbb{R}^{2d}) \text{ which generate a Gabor frame } \{\pi(Lk)g\}_{k \in \mathbb{Z}^{2d}}\big\}. \tag{1.4}$$

The set F_{L^2} is not open. For example, the characteristic function $g = 1_{[0,1]^d}$ generates a frame for $L = \left(\begin{smallmatrix} aI & 0 \\ 0 & bI \end{smallmatrix}\right)$ with $a = b = 1$ but not with any $a > 1$. Even worse, when this g is fixed and a and b are varying, one is led to a section of F_{L^2} which exhibits the strange picture of Janssen's tie mentioned above. In contrast, our first main result, Thm. 1.1(i), is a positive statement for $F_{M^1_s}$, where $M^1_s(\mathbb{R}^d)$ is the modulation space defined precisely in Sec. 2. Thus, the desired stability of Gabor frames under a perturbation of the lattice constants, which in general fails for atoms from $L^2(\mathbb{R}^d)$, will be obtained in a general form for atoms from a suitable subspace of $L^2(\mathbb{R}^d)$.

As our second main result, in Thm. 1.1(ii) we will prove that for a Gabor atom in the appropriate window class, the canonical dual atom depends continuously on the atom and the lattice parameters.

Let $\widetilde{g} = S^{-1}_{g,L} g$ denote the canonical dual Gabor atom. Then our first main theorem is as follows.

Theorem 1.1. *Let $s \geq 0$.*

(i) *The set $F_{M^1_s}$ is open in $M^1_s(\mathbb{R}^d) \times GL(\mathbb{R}^{2d})$.*

(ii) *The mapping $(g, L) \mapsto \widetilde{g}$ is continuous from $F_{M^1_s}$ into $M^1_s(\mathbb{R}^d)$.*

As a corollary we obtain an analogous result for the Schwartz space $\mathscr{S}(\mathbb{R})$.

Corollary 1.2. (i) *The set $F_{\mathscr{S}}$ is open in $\mathscr{S}(\mathbb{R}^d) \times GL(\mathbb{R}^{2d})$.*
(ii) *The mapping $(g, L) \mapsto \widetilde{g}$ is continuous from $F_{\mathscr{S}}$ into $\mathscr{S}(\mathbb{R}^d)$.*

Similarly to the set of "good" lattice constants in Question I, we have that the set $F_{M^1_s}$ being open is equivalent to a stability statement for the given class of Gabor frames. In fact, Thm. 1.1(i) implies a very general form of stability, where both the Gabor atom and the time-frequency lattice are allowed to vary at the same time. We note that Question I above is settled in a positive way by Thm 1.1(i) and Corollary 1.2(i). Part (ii) of the theorem and the corollary, respectively, yield positive answers to Question II. The details will be given in the examples in Sec. 5.

Next, we are concerned with complementary statements for Gabor systems which generate a Riesz basic sequence, i.e., a Riesz basis for its closed linear span. For a subspace $X \subseteq L^2(\mathbb{R}^d)$ define the set

$$R_X = \big\{(g, L) \in X \times GL(\mathbb{R}^{2d}) \text{ which generate a Gabor Riesz basic sequence } \{\pi(Lk)g\}_{k\in\mathbb{Z}^{2d}}\big\}. \tag{1.5}$$

As with the set of Gabor frames F_{L^2}, we have that the set of Gabor Riesz basic sequences R_{L^2} is not open. Yet from the positive results for $F_{M^1_s}$ we obtain the following positive result for $R_{M^1_s}$.

For a Gabor Riesz basic sequence generated by (g, L), let $\widetilde{g}^R$ denote the biorthogonal atom, which satisfies

$$\langle \pi(Lk)g, \pi(Ll)\widetilde{g}^R\rangle = \delta_{k,l}, \qquad k, l \in \mathbb{Z}^{2d}. \tag{1.6}$$

Then our second main theorem is a complement to Thm. 1.1, as follows.

Theorem 1.3. *Let $s \geq 0$.*
(i) *The set $R_{M^1_s}$ is open in $M^1_s(\mathbb{R}^d) \times GL(\mathbb{R}^{2d})$.*
(ii) *The mapping $(g, L) \mapsto \widetilde{g}^R$ is continuous from $R_{M^1_s}$ into $M^1_s(\mathbb{R}^d)$.*

Again we obtain the corresponding result for the Schwartz space $\mathscr{S}(\mathbb{R})$.

Corollary 1.4. (i) *The set $R_{\mathscr{S}}$ is open in $\mathscr{S}(\mathbb{R}^d) \times GL(\mathbb{R}^{2d})$.*
(ii) *The mapping $(g, L) \mapsto \widetilde{g}^R$ is continuous from $R_{\mathscr{S}}$ into $\mathscr{S}(\mathbb{R}^d)$.*

Observe that all these results hold for arbitrary time-frequency lattices in any dimension, including non-symplectic lattices in particular.

The results stated in this paper are formulated for modulation spaces with polynomial weights, yet they can be proved for modulation spaces with more general (submultiplicative [25, Def. 11.1.1]) weight functions, such as subexponential weights, cf. [25, Sec. 12.1]. We restrict the presentation here to the polynomial weights in order to include more specific estimates at some places, as will be pointed out in the proof of Lemma 2.2(i). By using polynomial weights we also obtain the desired extension to the Schwartz space.

As an application of our results, we mention that they are crucial in order to prove stability and convergence properties for Gabor multipliers when the time-frequency lattice is varying, see [18, Sec. 3.6].

We outline the structure of the present paper. The main goal of Sec. 2 is to obtain results for operators defined as series of time-frequency shifts. Operators of this type are closely connected with the twisted convolution over sequence spaces. The first lemma is an independent statement concerned with dilations in a class of

Wiener amalgam spaces. In Sec. 3 the previous results are combined and applied to Gabor frame operators. This application is based on a general form of the Janssen representation. More precisely, the Janssen representation identifies a Gabor frame operator as a series of time-frequency shifts. As a consequence, in this section we are able to prove two key theorems (Thm. 3.6, Thm. 3.8), which are crucial for obtaining the main result, that is, the proof of Thm. 1.1. Extensions are obtained in Sec. 4 which, in particular, yield the proofs of Thm. 1.3, Corollary 1.2 and Corollary 1.4. Finally, in Sec. 5 we add concluding remarks and examples.

2. Preliminary results

2.1. **Dilation in Wiener amalgam spaces.** For a vector $x \in \mathbb{R}^n$ let $\|x\|$ denote the Euclidean norm. For $s \in \mathbb{R}$ let $\ell^1_s(\mathbb{Z}^n)$ denote the Banach space of complex sequences on $\mathbb{Z}^n$ generated by the norm

$$\|\alpha\|_{\ell^1_s} = \sum_{k \in \mathbb{Z}^n} |\alpha(k)| \, (1 + \|k\|)^s. \tag{2.1}$$

The Wiener amalgam space $W(C_0, \ell^1_s)(\mathbb{R}^n)$ is the Banach space of continuous functions on $\mathbb{R}^n$ for which the following norm is finite:

$$\|f\|_{W(C_0,\ell^1_s)} = \sum_{k \in \mathbb{Z}^n} \max_{\rho \in [0,1]^n} \Big\{ |f(k+\rho)| \Big\} (1 + \|k\|)^s. \tag{2.2}$$

Let $GL(\mathbb{R}^n)$ denote the group of invertible real $n \times n$ matrices with the norm

$$\|L\| = \sup_{\|x\|=1} \|Lx\|. \tag{2.3}$$

Since all matrix norms are equivalent, we have $L \to L_0$ in $GL(\mathbb{R}^n)$ if and only if the matrices L converge to L_0 entry by entry.

Let D_L denote the dilation by $L \in GL(\mathbb{R}^n)$,

$$D_L f(x) = f(L^{-1}x). \tag{2.4}$$

We show that on the Wiener amalgam spaces defined above the dilation is jointly continuous in its argument and the matrix L.

Lemma 2.1. *Let $s \geq 0$ be given. The mapping $(f, L) \mapsto D_L f$ is continuous from $W(C_0, \ell^1_s)(\mathbb{R}^n) \times GL(\mathbb{R}^n)$ into $W(C_0, \ell^1_s)(\mathbb{R}^n)$.*

Proof. Let $(f, L) \to (f_0, L_0)$ in $W(C_0, \ell^1_s)(\mathbb{R}^n) \times GL(\mathbb{R}^n)$. Without loss of generality we assume that there are constants $R_1, R_2 > 0$ such that the convergence $L \to L_0$ takes place in the set Q defined by

$$Q = Q_{R_1,R_2} = \big\{ L \in GL(\mathbb{R}^n) \colon \|L\| \leq R_1, \|L^{-1}\| \leq R_2 \big\}. \tag{2.5}$$

For any $R > 0$ the expression

$$\|f\|^{(R)}_{W(C_0,\ell^1_s)} = \int_{\mathbb{R}^n} \max_{\|\rho\| \leq R} \Big\{ |f(x+\rho)| \Big\} \, (1 + \|x\|)^s dx \tag{2.6}$$

is an equivalent norm on $W(C_0,\ell^1_s)(\mathbb{R}^n)$ (a proof for the unweighted case $s=0$ is given in [25, Lemma 6.1.1(b)]). Hence, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sup_{L\in Q} \max_{\|\rho\|\le 1} \Big\{ |f(L^{-1}x + L^{-1}\rho)| \Big\} \, (1+\|x\|)^s dx \\
&\le R_1 \int_{\mathbb{R}^n} \sup_{L\in Q} \max_{\|\rho\|\le 1} \Big\{ |f(x + L^{-1}\rho)| \Big\} \, (1+\|Lx\|)^s dx \\
&\le R_1 \int_{\mathbb{R}^n} \max_{\|\rho\|\le R_2} \Big\{ |f(x+\rho)| \Big\} \, \sup_{L\in Q} (1+\|Lx\|)^s dx \\
&\le R_1(1+R_1)^s \int_{\mathbb{R}^n} \max_{\|\rho\|\le R_2} \Big\{ |f(x+\rho)| \Big\} \, (1+\|x\|)^s dx \\
&= R_1(1+R_1)^s \|f\|^{(R_2)}_{W(C_0,\ell^1_s)} \le C\|f\|^{(1)}_{W(C_0,\ell^1_s)}
\end{aligned} \tag{2.7}$$

for some constant $C>0$, and conclude that

$$\begin{aligned}
\sup_{L\in Q} \|D_L f\|^{(1)}_{W(C_0,\ell^1_s)} &= \sup_{L\in Q} \int_{\mathbb{R}^n} \max_{\|\rho\|\le 1} \Big\{ |f(L^{-1}x + L^{-1}\rho)| \Big\} \, (1+\|x\|)^s dx \\
&\le \int_{\mathbb{R}^n} \sup_{L\in Q} \max_{\|\rho\|\le 1} \Big\{ |f(L^{-1}x + L^{-1}\rho)| \Big\} \, (1+\|x\|)^s dx \\
&\le C\|f\|^{(1)}_{W(C_0,\ell^1_s)}.
\end{aligned} \tag{2.8}$$

In the following we use the Lebesgue theorem of dominated convergence in the following form: If $\int_{\mathbb{R}^n} \sup_{L\to L_0} |h(L,x)| dx < \infty$, then

$$\lim_{L\to L_0} \int_{\mathbb{R}^n} h(L,x) dx = \int_{\mathbb{R}^n} \lim_{L\to L_0} h(L,x) dx. \tag{2.9}$$

Using (2.7) with $f=f_0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sup_{L\to L_0} \max_{\|\rho\|\le 1} \Big\{ |f_0(L^{-1}(x+\rho)) - f_0(L_0^{-1}(x+\rho))| \Big\} \, (1+\|x\|)^s dx \\
&\le 2 \int_{\mathbb{R}^n} \sup_{L\in Q} \max_{\|\rho\|\le 1} \Big\{ |f_0(L^{-1}(x+\rho))| \Big\} \, (1+\|x\|)^s \\
&\le 2C\|f_0\|^{(1)}_{W(C_0,\ell^1_s)} < \infty.
\end{aligned} \tag{2.10}$$

Since f_0 is uniformly continuous on compact sets, we have, for each $x\in\mathbb{R}^n$,

$$\max_{\|\rho\|\le 1} \Big\{ |f_0(L^{-1}(x+\rho)) - f_0(L_0^{-1}(x+\rho))| \Big\} \to 0. \tag{2.11}$$

Hence by dominated convergence we obtain

$$\begin{aligned}
&\lim_{L\to L_0} \|D_L f_0 - D_{L_0} f_0\|^{(1)}_{W(C_0,\ell^1_s)} \\
&= \lim_{L\to L_0} \int_{\mathbb{R}^n} \max_{\|\rho\|\le 1} \Big\{ |f_0(L^{-1}(x+\rho)) - f_0(L_0^{-1}(x+\rho))| \Big\} \, (1+\|x\|)^s dx \\
&= \int_{\mathbb{R}^n} \lim_{L\to L_0} \max_{\|\rho\|\le 1} \Big\{ |f_0(L^{-1}(x+\rho)) - f_0(L_0^{-1}(x+\rho))| \Big\} \, (1+\|x\|)^s dx \\
&= \int_{\mathbb{R}^n} 0 \; dx = 0.
\end{aligned} \tag{2.12}$$

Finally, from (2.8) and (2.12) we conclude that

$$
\begin{aligned}
&\|D_L f - D_{L_0} f_0\|^{(1)}_{W(C_0,\ell^1_s)} \\
&\le \|D_L f - D_L f_0\|^{(1)}_{W(C_0,\ell^1_s)} + \|D_L f_0 - D_{L_0} f_0\|^{(1)}_{W(C_0,\ell^1_s)} \qquad (2.13)\\
&\le C\|f - f_0\|^{(1)}_{W(C_0,\ell^1_s)} + \|D_L f_0 - D_{L_0} f_0\|^{(1)}_{W(C_0,\ell^1_s)} \to 0. \qquad \square
\end{aligned}
$$

2.2. **Modulation spaces and time-frequency shifts.** In the sequel, $\mathbb{R}^d$ is regarded as the time domain and $\mathbb{R}^{2d}$ is the time-frequency domain. Making use of the time-frequency shifts $\pi(\lambda)$, $\lambda = (x, \omega) \in \mathbb{R}^{2d}$, as defined in (1.1), we define the short time Fourier transform

$$V_g f(\lambda) = \langle f, \pi(\lambda) g\rangle = \int_{\mathbb{R}^d} f(t)\, \overline{g(t-x)}\, e^{-2\pi i \omega t}\, dt, \qquad (2.14)$$

$f, g \in L^2(\mathbb{R}^d)$. These operators extend (for g in the Schwartz space $\mathscr{S}(\mathbb{R}^d)$) to f in $\mathscr{S}'(\mathbb{R}^d)$, the space of tempered distributions [25, p. 41].

Let $\phi(x) = e^{-\pi\|x\|^2}$ denote the Gaussian function. For $1 \le p \le \infty$ and $s \in \mathbb{R}$ the modulation space $M^p_s(\mathbb{R}^d) = M^{p,p}_{v_s}(\mathbb{R}^d)$ with polynomial weight $v_s(\lambda) = (1 + \|\lambda\|)^s$ is defined as the Banach space (of tempered distributions) generated by the norm

$$\|f\|_{M^p_s} = \|(V_\phi f) v_s\|_{L^p} = \Big(\int_{\mathbb{R}^{2d}} |V_\phi f(\lambda)|^p\, (1 + \|\lambda\|)^{sp}\, d\lambda\Big)^{1/p} \qquad (2.15)$$

with the usual modification for $p = \infty$. See [25, Chapter 11] for details on modulation spaces.

The space $M^p_s(\mathbb{R}^d)$ is invariant under the Fourier transform, and it is dense in $\mathscr{S}'(\mathbb{R}^d)$. Moreover, $\mathscr{S}(\mathbb{R}^d)$ is dense in $M^p_s(\mathbb{R}^d)$ for $1 \le p < \infty$. When $s = 0$ we omit the subscript s. We have $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$ with equivalent norms, and we note the continuous embeddings

$$
\begin{aligned}
&\mathscr{S}(\mathbb{R}^d) \hookrightarrow M^1(\mathbb{R}^d) \hookrightarrow M^p(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \qquad \text{if } 1 \le p \le 2,\\
&L^2(\mathbb{R}^d) \hookrightarrow M^p(\mathbb{R}^d) \hookrightarrow M^\infty(\mathbb{R}^d) \hookrightarrow \mathscr{S}'(\mathbb{R}^d) \qquad \text{if } 2 \le p \le \infty.
\end{aligned} \qquad (2.16)
$$

The space $M^1(\mathbb{R}^d)$ is sometimes known as Feichtinger's algebra [25, p. 246]. It was introduced in [13] as a Segal algebra (defined over arbitrary locally compact abelian groups) under the name $S_0(\mathbb{R}^d)$, and was studied in connection with Gabor analysis in [21]. In [25, Sec. 12.1], the space $M^1(\mathbb{R}^d)$ and its weighted versions are identified as the appropriate window classes for general time-frequency analysis. Specifically, we have

$$\mathscr{S}(\mathbb{R}^d) \hookrightarrow M^1_s(\mathbb{R}^d) \hookrightarrow M^1(\mathbb{R}^d) \qquad \text{for } s \ge 0, \qquad (2.17)$$

cf. equation (4.1) of the present paper.

Let $\mathcal{L}(B)$ denote the space of bounded linear operators on a Banach space B. The space $\mathcal{L}(B)$ is a Banach space with the norm

$$\|T\|_{\mathcal{L}(B)} = \sup_{\|f\|_B = 1} \|Tf\|_B. \qquad (2.18)$$

The modulation spaces and $L^p(\mathbb{R}^d)$ are invariant under time-frequency shifts.

Lemma 2.2. *Let $1 \le p \le \infty$ and $0 \le |t| \le s$.*

(i) *For $\lambda \in \mathbb{R}^{2d}$ the time-frequency shift operator $\pi(\lambda)$ is isometric on $L^p(\mathbb{R}^d)$ and $M^p(\mathbb{R}^d)$. Moreover, $\pi(\lambda)$ is bounded on $M_t^p(\mathbb{R}^d)$ and*

$$\|\pi(\lambda)\|_{\mathcal{L}(M_t^p)} \le (1 + \|\lambda\|)^s. \tag{2.19}$$

(ii) *The mapping $(\lambda, f) \mapsto \pi(\lambda)f$ is continuous from $\mathbb{R}^{2d} \times M_s^1(\mathbb{R}^d)$ into $M_s^1(\mathbb{R}^d)$.*

Proof. First, for $\lambda = (x, \omega)$ we have

$$V_\phi(\pi(\lambda)f) = \pi(\mu)V_\phi f \tag{2.20}$$

with $\mu = ((x, \omega), (0, -x)) \in \mathbb{R}^{4d}$ [25, Lemma 3.1.3]. Note that the time-frequency shifts π on the left and right sides of (2.20) are acting on two different dimensions: $\mathbb{R}^{2d}$ and $\mathbb{R}^{4d}$, respectively.

(i) The Lebesgue measure is invariant under translation T_x, $x \in \mathbb{R}^d$, so

$$\|\pi(\lambda)f\|_{L^p} = \|e^{2\pi i\omega t}T_x f\|_{L^p} = \|f\|_{L^p}, \tag{2.21}$$

and the isometry of time-frequency shifts on $L^p(\mathbb{R}^d)$ follows. Next, from (2.20) and the isometry on $L^p(\mathbb{R}^{2d})$ we obtain the isometry on $M^p(\mathbb{R}^d)$ as

$$\|\pi(\lambda)f\|_{M^p} = \|V_\phi(\pi(\lambda)f)\|_{L^p} = \|\pi(\mu)V_\phi f\|_{L^p} = \|V_\phi f\|_{L^p} = \|f\|_{M^p}. \tag{2.22}$$

Finally, the relation (2.19), after including a constant $C > 0$ on the right hand side, holds for more general weights [25, Thm. 11.3.5(b)]. Tracing back the proof of this theorem to the proof of [25, Lemma 11.1.1(b)], we find that the constant can be removed for the particular weights that we consider. More precisely, if $0 \le |t| \le s$, then the weight v_t is v_s-moderate [25, Def. 11.1.1, Eq. (11.2)] with constant $C = 1$.

(ii) Let $(\lambda, f) \to (\lambda_0, f_0)$ in $\mathbb{R}^{2d} \times M_s^1(\mathbb{R}^d)$. Using (2.20), we obtain

$$\begin{aligned} \|\pi(\lambda)f_0 - \pi(\lambda_0)f_0\|_{M_s^1} &= \|V_\phi\big(\pi(\lambda)f_0 - \pi(\lambda_0)f_0\big)\ v_s\|_{L^1} \\ &= \|\big(\pi(\mu) - \pi(\mu_0)\big)V_\phi f_0\ v_s\|_{L^1} \to 0, \end{aligned} \tag{2.23}$$

since the mapping $\lambda \mapsto \mu$ is continuous from $\mathbb{R}^{2d}$ into $\mathbb{R}^{4d}$ and the mapping $\mu \to \pi(\mu)F_0$ is continuous from $\mathbb{R}^{4d}$ into $L^1_{v_s}(\mathbb{R}^{2d})$ with $F_0 = V_\phi f_0 \in L^1_{v_s}(\mathbb{R}^{2d})$. We conclude that

$$\begin{aligned} &\|\pi(\lambda)f - \pi(\lambda_0)f_0\|_{M_s^1} \\ &\le \|\pi(\lambda)\|_{\mathcal{L}(M_s^1)}\|f - f_0\|_{M_s^1} + \|\pi(\lambda)f_0 - \pi(\lambda_0)f_0\|_{M_s^1} \\ &\le (1 + \|\lambda\|)^s\|f - f_0\|_{M_s^1} + \|\pi(\lambda)f_0 - \pi(\lambda_0)f_0\|_{M_s^1} \to 0, \end{aligned} \tag{2.24}$$

and so the statement in (ii) is proved. □

2.3. **Operators defined as series of time-frequency shifts.** In the sequel we are concerned with operators defined as series of time-frequency shifts along a lattice $\Lambda = L\mathbb{Z}^{2d} \subseteq \mathbb{R}^{2d}$, $L \in GL(\mathbb{R}^{2d})$.

Definition 2.3. Whenever for a given $L \in GL(\mathbb{R}^{2d})$ and a sequence α over $\mathbb{Z}^{2d}$ the series described below is convergent (in a suitable sense) we will denote by $A_{\alpha,L}$ the operator

$$A_{\alpha,L} = \sum_{k \in \mathbb{Z}^{2d}} \alpha(k)\pi(Lk). \tag{2.25}$$

Example 2.4. For $L = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, $a, b > 0$, and a suitable sequence α over $\mathbb{Z}^2$ we have

$$A_{\alpha,L} f(t) = \sum_{(k_1,k_2)\in\mathbb{Z}^2} \alpha(k_1,k_2) e^{2\pi i b k_2 t} f(t - a k_1). \tag{2.26}$$

Operators of this type arise in Gabor analysis (see [11, 17, 22, 29, 31] and [25, Sec. 13.4]) due to the Janssen representation of Gabor frame operators. See Sec. 3 for details.

Lemma 2.5. *Let $1 \le p \le \infty$ and $0 \le |t| \le s$ be given. For $L \in GL(\mathbb{R}^{2d})$ and $\alpha \in \ell^1_s(\mathbb{Z}^{2d})$ the operator $A_{\alpha,L}$ is bounded on $L^p(\mathbb{R}^d)$ and $M^p_t(\mathbb{R}^d)$, with absolute convergence of the series, and we have:*
(i) $\|A_{\alpha,L}\|_{\mathcal{L}(L^p)} \le \|\alpha\|_{\ell^1}$, *and*
(ii) $\|A_{\alpha,L}\|_{\mathcal{L}(M^p_t)} \le (1 + \|L\|)^s \, \|\alpha\|_{\ell^1_s}$.

Proof. By Lemma 2.2(i) we have

$$\|\pi(Lk)\|_{\mathcal{L}(L^p)} = 1 \qquad \text{for all } k \in \mathbb{Z}^{2d}, \tag{2.27}$$

and

$$\|\pi(Lk)\|_{\mathcal{L}(M^p_t)} \le (1 + \|Lk\|)^s \le (1 + \|L\|)^s (1 + \|k\|)^s, \qquad k \in \mathbb{Z}^{2d}. \tag{2.28}$$

From (2.27) we conclude that

$$\begin{aligned} \|A_{\alpha,L}\|_{\mathcal{L}(L^p)} &= \Big\| \sum_{k\in\mathbb{Z}^{2d}} \alpha(k)\pi(Lk) \Big\|_{\mathcal{L}(L^p)} \\ &\le \sum_{k\in\mathbb{Z}^{2d}} |\alpha(k)| \, \|\pi(Lk)\|_{\mathcal{L}(L^p)} = \sum_{k\in\mathbb{Z}^{2d}} |\alpha(k)| = \|\alpha\|_{\ell^1}, \end{aligned} \tag{2.29}$$

and this is finite since $\|\alpha\|_{\ell^1} \le \|\alpha\|_{\ell^1_s}$ for $s \ge 0$. From (2.28), we therefore obtain

$$\begin{aligned} \|A_{\alpha,L}\|_{\mathcal{L}(M^p_t)} &\le \sum_{k\in\mathbb{Z}^{2d}} |\alpha(k)| \|\pi(Lk)\|_{\mathcal{L}(M^p_t)} \\ &\le (1 + \|L\|)^s \sum_{k\in\mathbb{Z}^{2d}} |\alpha(k)|(1 + \|k\|)^s = (1 + \|L\|)^s \, \|\alpha\|_{\ell^1_s}. \end{aligned} \tag{2.30}$$ □

Remark 2.6. An inspection of the proof of Lemma 2.5 yields that for $\alpha \in \ell^1(\mathbb{Z}^{2d})$ the operator $A_{\alpha,L}$ is bounded not only on $L^p(\mathbb{R}^d)$ and $M^p(\mathbb{R}^d)$ but indeed on any Banach space B on which time-frequency shifts act isometrically (so that (2.27) and hence (2.29) hold analogously for the norm on $\mathcal{L}(B)$), e.g., solid Banach spaces of functions which are isometrically translation invariant, such as Lorentz spaces.

By Lemma 2.5, for a fixed $L \in GL(\mathbb{R}^{2d})$ the linear mapping $\alpha \mapsto A_{\alpha,L}$ is continuous from $\ell^1(\mathbb{Z}^{2d})$ into $\mathcal{L}(L^p(\mathbb{R}^d))$ and from $\ell^1_s(\mathbb{Z}^{2d})$ into $\mathcal{L}(M^p_t(\mathbb{R}^d))$ for each $1 \le p \le \infty$ and $0 \le |t| \le s$. However, the norm continuity breaks down if L is allowed to vary. In fact, if $L_1 \ne L_2$ in $GL(\mathbb{R}^{2d})$, then $L_1 k_0 \ne L_2 k_0$ for some $k_0 \in \mathbb{Z}^{2d}$. Hence with

$$\alpha(k) = \delta_{k_0,k} = \begin{cases} 1 & \text{if } k = k_0, \\ 0 & \text{otherwise}, \end{cases} \tag{2.31}$$

we find for any p in $1 \le p \le \infty$ that

$$\|A_{\alpha,L_1} - A_{\alpha,L_2}\|_{\mathcal{L}(L^p)} = \|\pi(L_1 k_0) - \pi(L_2 k_0)\|_{\mathcal{L}(L^p)} = 2, \tag{2.32}$$

and in the same way $\|A_{\alpha,L_1}-A_{\alpha,L_2}\|_{\mathcal{L}(M^p_t)}=2$. Nevertheless, we have the following results.

Proposition 2.7. *Let $1\le p\le\infty$ and $0\le|t|\le s$.*

(i) *At multiples of the unit vector $\alpha(k)=c\delta_{0,k}$, $c\in\mathbb{C}$, the mapping $(\alpha,L)\mapsto A_{\alpha,L}$ is continuous from $\ell^1(\mathbb{Z}^{2d})\times GL(\mathbb{R}^{2d})$ into $\mathcal{L}(L^p(\mathbb{R}^d))$ and, in addition, from $\ell^1_s(\mathbb{Z}^{2d})\times GL(\mathbb{R}^{2d})$ into $\mathcal{L}(M^p_t(\mathbb{R}^d))$.*

(ii) *The mapping $(\alpha,L,f)\mapsto A_{\alpha,L}f$ is continuous from $\ell^1_s(\mathbb{Z}^{2d})\times GL(\mathbb{R}^{2d})\times M^1_s(\mathbb{R}^d)$ into $M^1_s(\mathbb{R}^d)$.*

Proof. Let $(\alpha,L,f)\to(\alpha_0,L_0,f_0)$ in $\ell^1_s(\mathbb{Z}^{2d})\times GL(\mathbb{R}^{2d})\times M^p_t(\mathbb{R}^d)$.

(i) If $\alpha_0(k)=c\delta_{0,k}$ for some $c\in\mathbb{C}$, then

$$\begin{aligned}A_{\alpha_0,L}-A_{\alpha_0,L_0}&=\sum_{k\in\mathbb{Z}^{2d}}c\delta_{0,k}\pi(Lk)-\sum_{k\in\mathbb{Z}^{2d}}c\delta_{0,k}\pi(L_0k)\\&=c\,\mathrm{Id}-c\,\mathrm{Id}=0.\end{aligned}\tag{2.33}$$

Hence, from Lemma 2.5(i) we conclude that

$$\begin{aligned}\|A_{\alpha,L}-A_{\alpha_0,L_0}\|_{\mathcal{L}(L^p)}&\le\|A_{\alpha,L}-A_{\alpha_0,L}\|_{\mathcal{L}(L^p)}+\|A_{\alpha_0,L}-A_{\alpha_0,L_0}\|_{\mathcal{L}(L^p)}\\&\le\|\alpha-\alpha_0\|_{\ell^1}+0\to0\end{aligned}\tag{2.34}$$

and in a similar way from Lemma 2.5(ii) we obtain

$$\|A_{\alpha,L}-A_{\alpha_0,L_0}\|_{\mathcal{L}(M^p_t)}\le(1+\|L\|)^s\|\alpha-\alpha_0\|_{\ell^1_s}+0\to0,\tag{2.35}$$

so statement (i) is proved.

(ii) Without loss of generality we assume that there is a constant $R>0$ such that the convergence $L\to L_0$ takes place in the set Q defined by

$$Q=Q_R=\big\{L\in GL(\mathbb{R}^{2d})\colon\|L\|\le R\big\}.\tag{2.36}$$

We will use the following discrete version of Lebesgue dominated convergence: If $\sum_{k\in\mathbb{Z}^n}\sup_{L\to L_0}|h(L,k)|<\infty$, then

$$\lim_{L\to L_0}\sum_{k\in\mathbb{Z}^n}h(L,k)=\sum_{k\in\mathbb{Z}^n}\lim_{L\to L_0}h(L,k).\tag{2.37}$$

By Lemma 2.2(i) as used in (2.28) with $p=1$ and $t=s$, we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^{2d}}\sup_{L\to L_0}|\alpha_0(k)|\,\|\pi(Lk)f_0-\pi(L_0k)f_0\|_{M^1_s}\\&\le2\sum_{k\in\mathbb{Z}^{2d}}|\alpha_0(k)|\,\sup_{L\in Q}\|\pi(Lk)\|_{\mathcal{L}(M^1_s)}\,\|f_0\|_{M^1_s}\\&\le2(1+R)^s\sum_{k\in\mathbb{Z}^{2d}}|\alpha_0(k)|\,(1+\|k\|)^s\,\|f_0\|_{M^1_s}\\&=2(1+R)^s\,\|\alpha_0\|_{\ell^1_s}\,\|f_0\|_{M^1_s}<\infty.\end{aligned}\tag{2.38}$$

By Lemma 2.2(ii) we have, for each $k\in\mathbb{Z}^{2d}$,

$$\|\pi(Lk)f_0-\pi(L_0k)f_0\|_{M^1_s}\to0.\tag{2.39}$$

So by dominated convergence we obtain

$$\begin{aligned}
&\lim_{L\to L_0} \|A_{\alpha_0,L}f_0 - A_{\alpha_0,L_0}f_0\|_{M^1_s} \\
&= \lim_{L\to L_0} \Big\| \sum_{k\in\mathbb{Z}^{2d}} \alpha_0(k)\pi(Lk)f_0 - \sum_{k\in\mathbb{Z}^{2d}} \alpha_0(k)\pi(L_0k)f_0 \Big\|_{M^1_s} \\
&\le \lim_{L\to L_0} \sum_{k\in\mathbb{Z}^{2d}} |\alpha_0(k)|\ \|\pi(Lk)f_0 - \pi(L_0k)f_0\|_{M^1_s} \\
&= \sum_{k\in\mathbb{Z}^{2d}} \lim_{L\to L_0} |\alpha_0(k)|\ \|\pi(Lk)f_0 - \pi(L_0k)f_0\|_{M^1_s} = \sum_{k\in\mathbb{Z}^{2d}} 0 = 0.
\end{aligned} \tag{2.40}$$

By Lemma 2.5(ii) with $p = 1$ and $t = s$, we have

$$\begin{aligned}
&\|A_{\alpha,L}f - A_{\alpha_0,L}f_0\|_{M^1_s} \\
&\le \|A_{\alpha-\alpha_0,L}f\|_{M^1_s} + \|A_{\alpha_0,L}(f - f_0)\|_{M^1_s} \\
&\le (1 + \|L\|)^s\, \|\alpha - \alpha_0\|_{\ell^1_s}\, \|f\|_{M^1_s} + (1 + \|L\|)^s\, \|\alpha_0\|_{\ell^1_s}\, \|f - f_0\|_{M^1_s} \to 0.
\end{aligned} \tag{2.41}$$

In combination with (2.40) we conclude that

$$\begin{aligned}
&\|A_{\alpha,L}f - A_{\alpha_0,L_0}f_0\|_{M^1_s} \\
&\le \|A_{\alpha,L}f - A_{\alpha_0,L}f_0\|_{M^1_s} + \|A_{\alpha_0,L}f_0 - A_{\alpha_0,L_0}f_0\|_{M^1_s} \to 0.
\end{aligned} \tag{2.42}$$

This proves statement (ii). □

2.4. **Twisted convolution.** For fixed L the operators $A_{\alpha,L}$ with $\alpha \in \ell^1_s(\mathbb{Z}^{2d})$ form a Banach algebra, and the composition of such operators corresponds to a twisted convolution of the sequences α; see [25, 29, 31] for the case $L = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, $a, b > 0$. The twisted convolution in a more general context is defined in [17, Corollary 7.7.9] and [29, final section]. We give a slightly modified and explicit definition for $L \in GL(\mathbb{R}^{2d})$.

Definition 2.8. For $L \in GL(\mathbb{R}^{2d})$ we define the L-twisted convolution of α and β in $\ell^1(\mathbb{Z}^{2d})$ to be

$$(\alpha \natural_L \beta)(k) = \sum_{l\in\mathbb{Z}^{2d}} \alpha(l)\, \beta(k - l)\, e^{2\pi i (k-l)^T L^T K L l}, \qquad k \in \mathbb{Z}^{2d}, \tag{2.43}$$

where $K = \left(\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right) \in GL(\mathbb{R}^{2d})$.

The following example coincides with the twisted convolution of [25, Eq. (13.22)].

Example 2.9. For $L = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, $a, b > 0$, and $\alpha, \beta \in \ell^1(\mathbb{Z}^2)$ we have

$$(\alpha \natural_L \beta)(k_1, k_2) = \sum_{(l_1,l_2)\in\mathbb{Z}^2} \alpha(l_1, l_2)\, \beta(k_1 - l_1, k_2 - l_2)\, e^{2\pi i a b (k_1 - l_1) l_2}. \tag{2.44}$$

Implicit in (2.43) we find the polarized version $(\lambda, \mu) \mapsto e^{2\pi i \mu^T K \lambda}$ of the symplectic form [25, Eq. (9.6)] on $\mathbb{R}^{2d}$. The symplectic form appears in the definition of the product for the Heisenberg group. More specifically, the time-frequency shift operators $\pi(\lambda)$ are a representation of the (polarized) Heisenberg group, and we have

$$\pi(\lambda)\pi(\mu) = e^{2\pi i \mu^T K \lambda}\, \pi(\lambda + \mu), \tag{2.45}$$

see [25, Eq. (9.1)].

As indicated above, the L-twisted convolution is associated with the composition of operators of the type $A_{\alpha,L}$ as follows.

Lemma 2.10. *Let $L \in GL(\mathbb{R}^{2d})$ and $\alpha, \beta \in \ell^1(\mathbb{Z}^{2d})$. Then*

$$A_{\alpha,L} A_{\beta,L} = A_{\alpha \natural_L \beta, L}. \tag{2.46}$$

Proof. Using (2.45), we calculate

$$\begin{aligned} A_{\alpha,L} A_{\beta,L} &= \sum_{l \in \mathbb{Z}^{2d}} \alpha(l) \pi(Ll) \sum_{k \in \mathbb{Z}^{2d}} \beta(k) \pi(Lk) \\ &= \sum_{k \in \mathbb{Z}^{2d}} \sum_{l \in \mathbb{Z}^{2d}} \alpha(l) \beta(k) \pi(Ll) \pi(Lk) \\ &= \sum_{k \in \mathbb{Z}^{2d}} \sum_{l \in \mathbb{Z}^{2d}} \alpha(l) \beta(k) e^{2\pi i (Lk)^T KLl} \pi(L(l+k)) \qquad (2.47) \\ &= \sum_{k \in \mathbb{Z}^{2d}} \sum_{l \in \mathbb{Z}^{2d}} \alpha(l) \beta(k-l) e^{2\pi i (L(k-l))^T KLl} \pi(Lk) = A_{\alpha \natural_L \beta, L}. \end{aligned}$$

□

As a key lemma we prove that the twisted convolution is a continuous mapping even when the parameter, the matrix L, is allowed to vary.

Lemma 2.11. *Let $s \geq 0$. The mapping $(\alpha, \beta, L) \mapsto \alpha \natural_L \beta$ is continuous from $\ell^1_s(\mathbb{Z}^{2d}) \times \ell^1_s(\mathbb{Z}^{2d}) \times GL(\mathbb{R}^{2d})$ into $\ell^1_s(\mathbb{Z}^{2d})$.*

Proof. Let $(\alpha, \beta, L) \to (\alpha_0, \beta_0, L_0)$ in $\ell^1(\mathbb{Z}^{2d}) \times \ell^1(\mathbb{Z}^{2d}) \times GL(\mathbb{R}^{2d})$. Since

$$|(\alpha \natural_L \beta)(k)| \leq (|\alpha| * |\beta|)(k), \;\; k \in \mathbb{Z}^{2d},$$

the boundedness properties of the ordinary convolution carry over to the L-twisted convolution. In particular, for any $L \in GL(\mathbb{R}^{2d})$, the space $\ell^1_s(\mathbb{Z}^{2d})$ under the L-twisted convolution is a Banach algebra:

$$\|\alpha \natural_L \beta\|_{\ell^1_s} \leq \|\alpha\|_{\ell^1_s} \|\beta\|_{\ell^1_s}. \tag{2.48}$$

Since the estimate (2.48) is independent of L, we have that $(\alpha, \beta, L) \to (\alpha_0, \beta_0, L_0)$ implies

$$\|\alpha \natural_L \beta - \alpha_0 \natural_L \beta_0\|_{\ell^1_s} \to 0, \tag{2.49}$$

independently of L.

Next, we will use the discrete version of the Lebesgue theorem of dominated convergence as formulated in the proof of Prop. 2.7(ii), Eq. (2.37). We have

$$\begin{aligned} &\sum_{(k,l) \in \mathbb{Z}^{4d}} \sup_{L \to L_0} |\alpha_0(l)| \, |\beta_0(k-l)| \\ &\qquad\qquad \times |e^{2\pi i (k-l)^T L^T KLl} - e^{2\pi i (k-l)^T L_0{}^T KL_0 l}| \, (1 + \|k\|)^s \\ &\leq 2 \sum_{(k,l) \in \mathbb{Z}^{4d}} |\alpha_0(l)| \, |\beta_0(k-l)| \, (1 + \|k\|)^s \qquad (2.50) \\ &= 2 \sum_{k \in \mathbb{Z}^{2d}} (|\alpha_0| * |\beta_0|)(k) \, (1 + \|k\|)^s \\ &= 2 \| \, |\alpha_0| * |\beta_0| \, \|_{\ell^1_s} \leq 2 \|\alpha_0\|_{\ell^1_s} \|\beta_0\|_{\ell^1_s} < \infty. \end{aligned}$$

Note that $L \to L_0$ implies that, for each $(k,l) \in \mathbb{Z}^{4d}$,

$$|e^{2\pi i(k-l)^T L^T KLl} - e^{2\pi i(k-l)^T L_0{}^T KL_0 l}| \to 0. \tag{2.51}$$

Hence by dominated convergence we obtain

$$\begin{aligned}
&\lim_{L\to L_0} \|\alpha_0 \natural_L \beta_0 - \alpha_0 \natural_{L_0} \beta_0\|_{\ell^1_s} \\
&= \lim_{L\to L_0} \sum_{k\in\mathbb{Z}^{2d}} \Big| \sum_{l\in\mathbb{Z}^{2d}} \alpha_0(l)\,\beta_0(k-l) \\
&\qquad\qquad \times (e^{2\pi i(k-l)^T L^T KLl} - e^{2\pi i(k-l)^T L_0{}^T KL_0 l})\Big| (1+\|k\|)^s \\
&\le \lim_{L\to L_0} \sum_{(k,l)\in\mathbb{Z}^{4d}} |\alpha_0(l)|\;|\beta_0(k-l)| \\
&\qquad\qquad \times |e^{2\pi i(k-l)^T L^T KLl} - e^{2\pi i(k-l)^T L_0{}^T KL_0 l}|\,(1+\|k\|)^s \\
&= \sum_{(k,l)\in\mathbb{Z}^{4d}} \lim_{L\to L_0} |\alpha_0(l)|\;|\beta_0(k-l)| \\
&\qquad\qquad \times |e^{2\pi i(k-l)^T L^T KLl} - e^{2\pi i(k-l)^T L_0{}^T KL_0 l}|\,(1+\|k\|)^s \\
&= \sum_{(k,l)\in\mathbb{Z}^{4d}} 0 = 0.
\end{aligned} \tag{2.52}$$

Finally, from (2.49) and (2.52) we conclude that

$$\begin{aligned}
&\|\alpha \natural_L \beta - \alpha_0 \natural_{L_0} \beta_0\|_{\ell^1_s} \\
&\le \|\alpha \natural_L \beta - \alpha_0 \natural_L \beta_0\|_{\ell^1_s} + \|\alpha_0 \natural_L \beta_0 - \alpha_0 \natural_{L_0} \beta_0\|_{\ell^1_s} \to 0.
\end{aligned} \tag{2.53}$$

□

3. Gabor frames, and proof of Theorem 1.1

3.1. **Gabor frames and the Janssen representation.** As references for Gabor analysis we mention [7, 10, 19, 20, 25, 30]. Let $S_{g,\gamma,L}$ denote the frame-type operator with analysis window $g \in L^2(\mathbb{R}^d)$, synthesis atom $\gamma \in L^2(\mathbb{R}^d)$ and time-frequency lattice $\Lambda = L\mathbb{Z}^{2d} \subseteq \mathbb{R}^{2d}$ generated by the matrix $L \in GL(\mathbb{R}^{2d})$. That is,

$$S_{g,\gamma,L} f = \sum_{k\in\mathbb{Z}^{2d}} \langle f, \pi(Lk)g\rangle \pi(Lk)\gamma, \qquad f \in L^2(\mathbb{R}^d), \tag{3.1}$$

cf. [25, p. 111]. In addition, $S_{g,L} = S_{g,g,L}$ will denote the Gabor frame operator for the Gabor system $\{\pi(Lk)g\}_{k\in\mathbb{Z}^{2d}}$ generated by (g, L). Recall that (g, L) generates a Gabor frame if $S_{g,L}$ is both bounded and invertible on $L^2(\mathbb{R}^d)$.

Example 3.1. For $g, \gamma \in L^2(\mathbb{R})$ and $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, $a, b > 0$, we have $\Lambda = L\mathbb{Z}^2$ with $L = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, and

$$S_{g,\gamma,L} f = \sum_{(k_1,k_2)\in\mathbb{Z}^2} \langle f, \pi(ak_1, bk_2)g\rangle\, \pi(ak_1, bk_2)\gamma, \tag{3.2}$$

with $\pi(ak_1, bk_2)f(t) = e^{2\pi i bk_2 t} f(t - ak_1)$.

Continuity properties of Gabor frame operators have been proved in [39] based on the decomposition of a Gabor frame operator into a series of weighted translations. This is now referred to as the Walnut representation [25, Thm. 6.3.2], [25, Sec. 7.1], see also [2]. We are interested in changing the time-frequency lattice, and will use another important representation, the decomposition into a series of time-frequency shifts. This was introduced in [31] and is now known as the Janssen representation [25, Sec. 7.2].

The original Janssen representation was discovered in [31] for product lattices, see [25, Thm. 7.2.1] and Example 3.4 below. With the metaplectic representation [25, Sec. 9.4] the extension to symplectic lattices [25, Corollary 9.4.5] follows easily. The general form for arbitrary lattices was developed in [17] and is proved in [21]. The key to the Janssen representation for arbitrary lattices is the adjoint lattice Λ° of a lattice Λ, defined in [17] and [21] by

$$\Lambda^\circ = \{\lambda^\circ \in \mathbb{R}^{2d} \colon \pi(\lambda^\circ)\pi(\lambda) = \pi(\lambda)\pi(\lambda^\circ) \text{ for all } \lambda \in \Lambda\}. \tag{3.3}$$

According to the definition this adjointness relation is involutive, i. e., $(\Lambda^\circ)^\circ = \Lambda$. The adjoint lattice Λ° is a rotation of the orthogonal lattice $\Lambda^\perp = (L^{-1})^T\mathbb{Z}^{2d}$ (also known as the annihilator, dual or reciprocal lattice), namely,

$$\Lambda^\circ = J\Lambda^\perp, \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in GL(\mathbb{R}^{2d}). \tag{3.4}$$

(The matrix J is related to the symmetric version of the symplectic form on $\mathbb{R}^{2d}$ in a similar way as the matrix K of Definition 2.8 is related to the polarized version, cf. the paragraph after Example 2.9.) In particular, the matrix $J(L^{-1})^T$ is a generator for Λ°. The generator of a lattice is not unique, and for purposes of symmetry we prefer the following choice.

Definition 3.2. For $L \in GL(\mathbb{R}^{2d})$ we define

$$L^\circ = J(L^{-1})^T J^T. \tag{3.5}$$

For example, this definition maps $L = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ to $L^\circ = \begin{pmatrix} 1/b & 0 \\ 0 & 1/a \end{pmatrix}$.

The coefficients for the Janssen representation are given by sampling the short-time Fourier transform $V_g\gamma$ on the adjoint lattice.

Definition 3.3. For $g, \gamma \in L^2(\mathbb{R}^d)$ and $L \in GL(\mathbb{R}^{2d})$, define

$$\alpha_{g,\gamma,L^\circ}(k) = |\det L|^{-1}\ V_g\gamma(L^\circ k), \qquad k \in \mathbb{Z}^{2d}. \tag{3.6}$$

We will abbreviate α_{g,g,L° as α_{g,L°.

With this notation the Janssen representation for general time-frequency lattices reads as follows (our notation of the frame-type operator differs from the original source [21]). For $g, \gamma \in L^2(\mathbb{R}^d)$ the frame-type operator $S_{g,\gamma,L}$ has the representation

$$S_{g,\gamma,L} = A_{\alpha_{g,\gamma,L^\circ},L^\circ} = \sum_{k\in\mathbb{Z}^{2d}} \alpha_{g,\gamma,L^\circ}(k)\,\pi(L^\circ k). \tag{3.7}$$

The following example is the original Janssen representation.

Example 3.4. Let $g, \gamma \in M^1(\mathbb{R})$ and $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, $a, b > 0$. Since $\Lambda = L\mathbb{Z}^2$ with $L = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, we have $L^\circ = \begin{pmatrix} 1/b & 0 \\ 0 & 1/a \end{pmatrix}$ and $\Lambda^\circ = \frac{1}{b}\mathbb{Z} \times \frac{1}{a}\mathbb{Z} = L^\circ\mathbb{Z}^2$, and hence

$$S_{g,\gamma,L} = \sum_{(k_1,k_2)\in\mathbb{Z}^2} \alpha_{g,\gamma,L^\circ}(k_1,k_2)\,\pi\big(\tfrac{1}{b}k_1, \tfrac{1}{a}k_2\big), \tag{3.8}$$

where

$$\alpha_{g,\gamma,L^\circ}(k_1,k_2) = \frac{1}{ab}\langle\gamma, \pi\big(\tfrac{1}{b}k_1, \tfrac{1}{a}k_2\big)g\rangle. \tag{3.9}$$

For general $g, \gamma \in L^2(\mathbb{R}^d)$ the frame-type operator $S_{g,\gamma,L}$ maps $M^1(\mathbb{R}^d)$ into $M^\infty(\mathbb{R}^d)$, and the series (3.7) applied to $f \in M^1(\mathbb{R}^d)$ converges unconditionally in $M^\infty(\mathbb{R}^d)$ [21, Thm. 3.5.11(iii)]. Under the additional assumption that g, γ are in $M^1(\mathbb{R}^d)$, the restriction of $V_g\gamma$ to arbitrary time-frequency lattices Λ is in $\ell^1(\mathbb{Z}^{2d})$ [21, Lemma 3.5.15]. Consequently, the operator is also bounded on $L^p(\mathbb{R}^d)$ and $M^p(\mathbb{R}^d)$, $1 \le p \le \infty$ (Lemma 2.5 with $s = 0$) and indeed on any Banach space on which time-frequency shifts act isometrically (Remark 2.6). Note that $M^1(\mathbb{R}^d)$ is the largest reasonable subspace of $L^2(\mathbb{R}^d)$ for which the Janssen coefficients have these good summability properties [21, Lemma 3.5.15].

If L is fixed, then the coefficients $\alpha_{g,\gamma,L^\circ}$ depend continuously in the $\ell^1_s(\mathbb{Z}^{2d})$-norm on g and γ in $M^1_s(\mathbb{R}^d)$; see [25, Corollary 12.1.12] for the case of product lattices. We next show the joint continuity including the lattice generator matrix L. In particular, for $g \in M^1_s(\mathbb{R}^d)$ the Janssen coefficients $\alpha_{g,\gamma,L^\circ}$ depend continuously in the $\ell^1_s(\mathbb{Z}^{2d})$-norm on the lattice parameters.

Lemma 3.5. *Let $s \ge 0$. The mapping $(g, \gamma, L) \to \alpha_{g,\gamma,L^\circ}$ is continuous from $M^1_s(\mathbb{R}^d) \times M^1_s(\mathbb{R}^d) \times GL(\mathbb{R}^{2d})$ into $\ell^1_s(\mathbb{Z}^{2d})$.*

Proof. First, the mapping

$$(g, \gamma) \mapsto V_g\gamma \tag{3.10}$$

is continuous from $M^1_s(\mathbb{R}^d) \times M^1_s(\mathbb{R}^d)$ into $W(C_0, \ell^1_s)(\mathbb{R}^{2d})$ [25, Eq. (12.15)]. Next, by Lemma 2.1 (with $n = 2d$) the mapping

$$(V_g\gamma, L) \mapsto D_L V_g\gamma \tag{3.11}$$

is continuous from $W(C_0, \ell^1_s)(\mathbb{R}^{2d}) \times GL(\mathbb{R}^{2d})$ into $W(C_0, \ell^1_s)(\mathbb{R}^{2d})$. Moreover, these Wiener amalgam spaces are defined in such a way that sampling is bounded from $W(C_0, \ell^1_s)(\mathbb{R}^{2d})$ into $\ell^1_s(\mathbb{Z}^{2d})$ [25, Prop. 11.1.4], i.e., the mapping

$$D_L V_g\gamma \mapsto (D_L V_g\gamma)|_{\mathbb{Z}^{2d}} \tag{3.12}$$

is continuous from $W(C_0, \ell^1_s)(\mathbb{R}^{2d})$ into $\ell^1_s(\mathbb{Z}^{2d})$. Finally, we have

$$\alpha_{g,\gamma,L^\circ} = |\det L|^{-1}(D_{(L^\circ)^{-1}} V_g\gamma)|_{\mathbb{Z}^{2d}} \tag{3.13}$$

by the definition of the coefficients $\alpha_{g,\gamma,L^\circ}$, so the statement follows since the mappings $L \mapsto \det L$ and $L \mapsto L^\circ$ are continuous for the respective topologies. ☐

3.2. **Two crucial results and proof of Theorem 1.1.** According to the Janssen representation (3.7) of the frame-type operator $S_{g,\gamma,L}$, the continuity results stated in Prop. 2.7 for operators of the form $A_{\alpha,L}$ apply, in particular, to Gabor frame-type operators. The following theorem is crucial for our main results.

Theorem 3.6. *Let $1 \le p \le \infty$ and $0 \le |t| \le s$.*

(i) *If $(g_0, L_0) \in M^1_s(\mathbb{R}^d) \times GL(\mathbb{R}^{2d})$ generates a Gabor frame with some dual Gabor atom $\gamma_0 \in M^1_s(\mathbb{R}^d)$, then at the point (g_0, γ_0, L_0) the mapping $(g, \gamma, L) \mapsto S_{g,\gamma,L}$ is continuous from $M^1(\mathbb{R}^d) \times M^1(\mathbb{R}^d) \times GL(\mathbb{R}^{2d})$ into $\mathcal{L}(L^p(\mathbb{R}^d))$ and, in addition, from $M^1_s(\mathbb{R}^d) \times M^1_s(\mathbb{R}^d) \times GL(\mathbb{R}^{2d})$ into $\mathcal{L}(M^p_t(\mathbb{R}^d))$.*

(ii) *The mapping $(g, \gamma, L, f) \mapsto S_{g,\gamma,L}f$ is continuous from $M^1_s(\mathbb{R}^d) \times M^1_s(\mathbb{R}^d) \times GL(\mathbb{R}^{2d}) \times M^1_s(\mathbb{R}^d)$ into $M^1_s(\mathbb{R}^d)$.*

Proof. (i) If g and γ are dual atoms, then $S_{g,\gamma,L} = \mathrm{Id}$. Therefore, the coefficients $\alpha_{g,\gamma,L^\circ}$ for the representation (3.7) of the frame-type operator $S_{g,\gamma,L}$ are given as

$$\alpha_{g,\gamma,L^\circ} = \delta_{k,0}, \qquad k \in \mathbb{Z}^{2d}. \tag{3.14}$$

Hence, in view of the representation (3.7), namely that $S_{g,\gamma,L} = A_{\alpha_{g,\gamma,L^\circ},L^\circ}$, the statement follows from Prop. 2.7(i).

(ii) According to Lemma 3.5, the coefficients $\alpha_{g,\gamma,L^\circ}$ are in $\ell^1_s(\mathbb{Z}^{2d})$ and depend continuously on g and γ. In addition, the mapping $L \to L^\circ$ is continuous on $GL(\mathbb{R}^{2d})$. Hence the statement follows from Prop. 2.7(ii). □

Remark 3.7. Statement (ii) in Thm. 3.6 includes the strong continuity of the Gabor frame-type operator $S_{g,\gamma,L}$ under a perturbation of the lattice parameters. We note that a strong continuity of this type appears in a different context in [32, Thm. 3.1, proof step (b)], where the L^2-convergence of the canonical tight Gabor atoms is described as the time-frequency lattice $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ tends to the critical limiting case $ab = 1$.

As a key result we obtain the stability of Gabor frames with Gabor atom in $M^1(\mathbb{R}^d)$ under a joint perturbation of the atom and the lattice parameter matrix.

Theorem 3.8. *Let $s \geq 0$ and assume that $(g_0, L_0) \in M^1_s(\mathbb{R}^d) \times GL(\mathbb{R}^{2d})$ generates a Gabor frame with some dual atom $\gamma_0 \in M^1_s(\mathbb{R}^d)$. Then there is a neighborhood U of (g_0, L_0) in $M^1_s(\mathbb{R}^d) \times GL(\mathbb{R}^{2d})$ such that any $(g, L) \in U$ generates a Gabor frame with some dual atom γ in $M^1_s(\mathbb{R}^d)$.*

Specifically, these dual atoms γ can be chosen such that the assignment $(g_0, L_0) \mapsto \gamma_0$ extends to a continuous mapping $(g, L) \mapsto \gamma$ from U into $M^1_s(\mathbb{R}^d)$.

Proof. If $(g_0, L_0) \in M^1_s(\mathbb{R}^d) \times GL(\mathbb{R}^{2d})$ generates a frame with dual atom γ_0, then $S_{g_0,\gamma_0,L_0} = \mathrm{Id}$. By Thm. 3.6(i) with $p = 1$ and $t = s$, there is a neighborhood U of (g_0, L_0) in $M^1_s(\mathbb{R}^d) \times GL(\mathbb{R}^{2d})$ such that

$$\|S_{g,\gamma_0,L} - \mathrm{Id}\|_{\mathcal{L}(M^1_s)} < 1 \qquad \text{for all } (g, L) \in U; \tag{3.15}$$

hence these operators $S_{g,\gamma_0,L}$ are invertible on $M^1_s(\mathbb{R}^d)$. In particular, the continuous mapping $(g, L) \mapsto S_{g,\gamma_0,L}$ goes into an open subset of the invertible operators on $M^1_s(\mathbb{R}^d)$. Since the inversion is a continuous mapping on this subset of the operator algebra, we obtain that the mapping

$$(g, L) \mapsto \gamma = S^{-1}_{g,\gamma_0,L}\gamma_0 \tag{3.16}$$

is continuous as well.

Next, a Gabor frame-type operator based on time-frequency shifts generated by L commutes with these time-frequency shifts (see [25, Eq. (5.25)]). So we have

$$S_{g,\gamma_0,L}\pi(Lk) = \pi(Lk)S_{g,\gamma_0,L} \tag{3.17}$$

for all $k \in \mathbb{Z}^{2d}$, and conclude that for all $f \in L^2(\mathbb{R}^d)$

$$\begin{aligned} S_{g,\gamma_0,L}S_{g,\gamma,L}f &= S_{g,\gamma_0,L} \sum_{k\in\mathbb{Z}^{2d}} \langle f, \pi(Lk)g\rangle \pi(Lk)\gamma \\ &= \sum_{k\in\mathbb{Z}^{2d}} \langle f, \pi(Lk)g\rangle \pi(Lk)S_{g,\gamma_0,L}\gamma \\ &= \sum_{k\in\mathbb{Z}^{2d}} \langle f, \pi(Lk)g\rangle \pi(Lk)\gamma_0 = S_{g,\gamma_0,L}f. \end{aligned} \tag{3.18}$$

Hence $S_{g,\gamma,L} = \mathrm{Id}$, so (g, L) generates a frame and γ is a dual atom. As stated above, γ depends continuously on $(g, L) \in U$, so the proof is complete. $\square$

If $(g, L) \in M^1_s(\mathbb{R}^d) \times GL(\mathbb{R}^{2d})$ generates a Gabor frame, then the canonical dual atom $\widetilde{g} = S^{-1}_{g,L} g$ belongs to $M^1_s(\mathbb{R}^d)$. For $L = \left(\begin{smallmatrix} aI & 0 \\ 0 & bI \end{smallmatrix}\right)$ with $ab \in \mathbb{Q}$ this is the main result of [15], and by the metaplectic representation the statement extends to symplectic lattices with $\det L \in \mathbb{Q}$ [25, Corollary 13.2.2]. Finally, the irrational case $\det L \notin \mathbb{Q}$, $L \in GL(\mathbb{R}^{2d})$, has been settled in [29]. This deep result is crucial for the proof of the main result of the present paper, Thm. 1.1, stated in the introduction.

Proof of Theorem 1.1. (i) Let $(g_0, L_0) \in F_{M^1_s}$. In other words, $(g_0, L_0) \in M^1_s(\mathbb{R}^d) \times GL(\mathbb{R}^{2d})$ generates a Gabor frame. As quoted above, the canonical dual atom $\widetilde{g}_0$ is contained in $M^1_s(\mathbb{R}^d)$. Therefore, according to Thm. 3.8 with $\gamma_0 = \widetilde{g}_0$, there is a neighborhood $U \subseteq M^1_s(\mathbb{R}^d) \times GL(\mathbb{R}^{2d})$ of (g_0, L_0) such that any $(g, L) \in U$ generates a Gabor frame. Hence $F_{M^1_s}$ is open.

(ii) Let $(g, L) \to (g_0, L_0)$ in $F_{M^1_s}$ and let $\widetilde{g}_0$ denote the canonical dual atom to (g_0, L_0). In view of the representation (3.7) of Gabor frame operators and the use of the L-twisted convolution for the composition of operators (Lemma 2.10), we have

$$\begin{aligned} &\|S_{\widetilde{g}_0,L} S_{g,L} - \mathrm{Id}\|_{\mathcal{L}(M^1_s)} \\ &= \|S_{\widetilde{g}_0,L} S_{g,L} - S_{\widetilde{g}_0,L_0} S_{g_0,L_0}\|_{\mathcal{L}(M^1_s)} \\ &= \Big\| \sum_{k\in\mathbb{Z}^{2d}} \alpha(k)\pi(L^\circ k) - \sum_{k\in\mathbb{Z}^{2d}} \alpha_0(k)\pi(L_0^\circ k) \Big\|_{\mathcal{L}(M^1_s)} \to 0, \end{aligned} \tag{3.19}$$

where $\alpha = \alpha_{g,L^\circ} \natural_{L^\circ} \alpha_{\widetilde{g}_0,L^\circ}$ and $\alpha_0 = \alpha_{g_0,L_0^\circ} \natural_{L_0^\circ} \alpha_{\widetilde{g}_0,L_0^\circ}$. Indeed, by Lemma 3.5 the coefficient sequences $\alpha_{g,L^\circ} \to \alpha_{g_0,L_0^\circ}$ and $\alpha_{\widetilde{g}_0,L^\circ} \to \alpha_{\widetilde{g}_0,L_0^\circ}$ converge in $\ell^1_s(\mathbb{Z}^{2d})$, so their L-twisted convolution $\alpha \to \alpha_0$ converges in $\ell^1_s(\mathbb{Z}^{2d})$ by Lemma 2.11. Hence (3.19) follows from Prop. 2.7(ii). Note that $S_{\widetilde{g}_0,L_0} S_{g_0,L_0} = \mathrm{Id}$ implies that $\alpha_0(k) = \delta_{0,k}$.

Next, from Lemma 2.5(ii) with $p = 1$ and $t = s$, we obtain the estimate

$$\|S_{\widetilde{g}_0,L}\|_{\mathcal{L}(M^1_s)} = \|A_{\alpha_{\widetilde{g}_0,L^\circ},L^\circ}\|_{\mathcal{L}(M^1_s)} \le (1 + \|L^\circ\|)^s \, \|\alpha_{\widetilde{g}_0,L^\circ}\|_{\ell^1_s} \tag{3.20}$$

(cf. [25, p. 289]). Moreover, in view of the convergence in (3.19) we have $S_{\widetilde{g}_0,L} S_{g,L} \to \mathrm{Id}$ in $\mathcal{L}(M^1_s(\mathbb{R}^d))$; hence also $(S_{\widetilde{g}_0,L} S_{g,L})^{-1} \to \mathrm{Id}$ in $\mathcal{L}(M^1_s(\mathbb{R}^d))$. As a consequence,

$$\begin{aligned} \|S^{-1}_{g,L}\|_{\mathcal{L}(M^1_s)} &= \|(S_{\widetilde{g}_0,L} S_{g,L})^{-1} S_{\widetilde{g}_0,L}\|_{\mathcal{L}(M^1_s)} \\ &\le \|(S_{\widetilde{g}_0,L} S_{g,L})^{-1}\|_{\mathcal{L}(M^1_s)} \, \|S_{\widetilde{g}_0,L}\|_{\mathcal{L}(M^1_s)} \\ &\le \|(S_{\widetilde{g}_0,L} S_{g,L})^{-1}\|_{\mathcal{L}(M^1_s)} \, (1 + \|L^\circ\|)^s \, \|\alpha_{\widetilde{g}_0,L^\circ}\|_{\ell^1_s} \\ &\to (1 + \|L_0^\circ\|)^s \, \|\alpha_{\widetilde{g}_0,L_0^\circ}\|_{\ell^1_s} < \infty, \end{aligned} \tag{3.21}$$

since $\alpha_{\widetilde{g}_0,L^\circ} \to \alpha_{\widetilde{g}_0,L_0^\circ}$ in $\ell^1_s(\mathbb{Z}^{2d})$ by Lemma 3.5. Without loss of generality we assume that the convergence $(g, L) \to (g_0, L_0)$ takes place close to (g_0, L_0), so that (3.21) implies

$$\sup_{(g,L)\to(g_0,L_0)} \|S^{-1}_{g,L}\|_{\mathcal{L}(M^1_s)} \le C < \infty. \tag{3.22}$$

Moreover, in view of Thm. 3.6(ii) we have that $(g, L) \to (g_0, L_0)$ yields

$$\|S_{g_0,L_0} \widetilde{g}_0 - S_{g,L} \widetilde{g}_0\|_{M^1_s} \to 0. \tag{3.23}$$

Finally, from (3.22) and (3.23) we conclude that

$$\begin{aligned}\|\widetilde{g}-\widetilde{g}_0\|_{M^1_s} &= \|S^{-1}_{g,L}g-\widetilde{g}_0\|_{M^1_s} \\ &\le \|S^{-1}_{g,L}\|_{\mathcal{L}(M^1_s)}\|g-S_{g,L}\widetilde{g}_0\|_{M^1_s} \\ &\le C\big(\|g-g_0\|_{M^1_s}+\|g_0-S_{g,L}\widetilde{g}_0\|_{M^1_s}\big) \\ &\le C\big(\|g-g_0\|_{M^1_s}+\|S_{g_0,L_0}\widetilde{g}_0-S_{g,L}\widetilde{g}_0\|_{M^1_s}\big)\to 0. \end{aligned} \tag{3.24}$$

□

4. Extensions, and proofs of Theorem 1.3, Corollary 1.2, and Corollary 1.4

4.1. **Riesz basic sequences.** A sequence of vectors in $L^2(\mathbb{R}^d)$ is called a Riesz basic sequence if it is a Riesz basis for its closed linear span. There is a duality between Gabor frames and Gabor Riesz basic sequences. More precisely, $\{\pi(Lk)g\}_{k\in\mathbb{Z}^{2d}}$ is a Gabor frame (with canonical dual atom $\widetilde{g}$) if and only if $\{\pi(L^\circ k)g\}_{k\in\mathbb{Z}^{2d}}$ is a Gabor Riesz basic sequence (with biorthogonal atom $\widetilde{g}^R=|\det L|^{-1}\widetilde{g}$) [21, Corollary 3.5.14].

For product lattices this result is implicit in [31], and it is proved and investigated in detail in [36]. It was first observed in the setting of finite cyclic groups in [40], and is now known as the Wexler-Raz, Janssen, Ron-Shen duality [25, Sec. 7.4]. Using the metaplectic representation, the statement extends to symplectic lattices [25, Corollary 9.4.7], and the general form has been developed in [17] and proved in the form given above in [21]. Accordingly, the conclusions of the present paper for Gabor frames imply the analogous results for Gabor Riesz basic sequences. The corresponding result has already been stated as Thm. 1.3 in the introduction, and is proved as follows.

Proof of Theorem 1.3. (i) The mapping $L\mapsto L^\circ$ yields a bijection between the sets $F_{M^1_s}$ and $R_{M^1_s}$. Since $L\mapsto L^\circ$ is continuous, the statement follows from Thm. 1.1(i).

(ii) The canonical dual of a Gabor frame and the corresponding biorthogonal atom differ only by a normalization factor $|\det L|$. Since this factor depends continuously on L, the statement follows from Thm. 1.1(ii). The proof of the subsequent Corollary 1.4 is given below, together with the proof of Corollary 1.2. □

4.2. **Schwartz space.** Another extension of our results is concerned with the Schwartz space $\mathscr{S}(\mathbb{R}^d)$. The Schwartz space coincides with the intersection of modulation spaces, i.e.,

$$\mathscr{S}(\mathbb{R}^d)=\bigcap_{s=0}^{\infty}M^1_s(\mathbb{R}^d) \tag{4.1}$$

[25, p. 254]. In fact, the norms of the modulation spaces $M^1_s(\mathbb{R}^d)$, $s=0,1,\dots$, are a nested system of semi-norms for $\mathscr{S}(\mathbb{R}^d)$. These facts are stated for the modulation spaces $M^\infty_s(\mathbb{R}^d)$ in [25, Prop. 11.3.1(d)] and [25, Corollary 11.2.6]. The identity (4.1) then follows from the continuous embeddings between modulation spaces $M^1_s(\mathbb{R}^d)\hookrightarrow M^\infty_s(\mathbb{R}^d)$ and $M^\infty_{s'}(\mathbb{R}^d)\hookrightarrow M^1_s(\mathbb{R}^d)$ for $s'>s+2d$ [25, Eq. (13.32)]. By using discrete equivalent norms on modulation spaces (see [25, Sec. 12.2]), such embeddings follow from the corresponding inclusions between weighted ℓ^p-spaces over $\mathbb{Z}^{2d}$.

Now our continuity results, stated for the modulation spaces $M^1_s(\mathbb{R}^d)$, extend to the Schwartz space $\mathscr{S}(\mathbb{R}^d)$ as follows. Note that if $g \in \mathscr{S}(\mathbb{R}^d)$ generates a Gabor frame, then the canonical dual atom $\widetilde{g}$ is in $\mathscr{S}(\mathbb{R}^d)$ as well [31], [25, Thm. 13.5.4].

Proof of Corollaries 1.2 and 1.4. (i) By Thm. 1.1(i) with $s = 0$ the set F_{M^1} is open in $M^1(\mathbb{R}^d) \times GL(\mathbb{R}^{2d})$. The set $F_{\mathscr{S}} = F_{M^1} \cap \big(\mathscr{S}(\mathbb{R}^d) \times GL(\mathbb{R}^{2d})\big)$ is a section of the open set F_{M^1}; hence it is open in the relative topology, since $\mathscr{S}(\mathbb{R}^d)$ is continuously embedded into $M^1(\mathbb{R}^d)$. This proves part (i) of Corollary 1.2. The corresponding statement for $R_{\mathscr{S}}$ in Corollary 1.4(i) follows analogously from Thm. 1.3(i).

(ii) As mentioned above, the topology on the Schwartz space may be derived from the system of norms of the modulation spaces $M^1_s(\mathbb{R}^d)$, $s = 0, 1, \dots$. Hence

$$\widetilde{g} \to \widetilde{g}_0 \text{ in } M^1_s(\mathbb{R}^d) \text{ for all } s = 0, 1, \dots \quad \Leftrightarrow \quad \widetilde{g} \to \widetilde{g}_0 \text{ in } \mathscr{S}(\mathbb{R}^d). \tag{4.2}$$

Therefore, statement (ii) of Corollary 1.2 follows from Thm. 1.1(ii). In the same way, Corollary 1.4(ii) follows from Thm. 1.3(ii). □

4.3. **The frame identity.** The composition of the Gabor analysis mapping with the dual synthesis mapping yields the frame identity. In other words, if (g_0, L_0) generates a Gabor frame and $\widetilde{g}_0 = S^{-1}_{g_0,L_0} g_0$ denotes the canonical dual atom, then

$$f = S_{g_0,\widetilde{g}_0,L_0} f \qquad \text{for all } f \in L^2(\mathbb{R}^d). \tag{4.3}$$

A more explicit way of writing this is the first identity of (1.3). Now the results of the present paper imply the following approximations of the frame identity. Observe that this approximation property holds for various function spaces simultaneously.

Theorem 4.1. *Let $(g_0, L_0) \in L^2(\mathbb{R}^d) \times GL(\mathbb{R}^{2d})$ generate a Gabor frame.*

(i) *If $g_0 \in M^1(\mathbb{R}^d)$, then for all $\varepsilon > 0$ there exist $\delta_1, \delta_2, \delta_3 > 0$ such that the assumption*

$$\begin{aligned} &\|g - g_0\|_{M^1} < \delta_1, \\ &\|L - L_0\| < \delta_2, \ \text{and} \\ &\|h - S^{-1}_{g_0,L_0} g_0\|_{M^1} < \delta_3 \ \text{or} \ \|h - S^{-1}_{g,L} g\|_{M^1} < \delta_3, \end{aligned} \tag{4.4}$$

implies that, for each $1 \le p \le \infty$,

$$\|f - S_{g,h,L} f\|_{L^p} \le \varepsilon \|f\|_{L^p} \qquad \text{for all } f \in L^p(\mathbb{R}^d). \tag{4.5}$$

(ii) *Let $s \ge 0$. If $g_0 \in M^1_s(\mathbb{R}^d)$, then for all $\varepsilon > 0$ there exist $\delta_1, \delta_2, \delta_3 > 0$ such that the assumption*

$$\begin{aligned} &\|g - g_0\|_{M^1_s} < \delta_1, \\ &\|L - L_0\| < \delta_2, \ \text{and} \\ &\|h - S^{-1}_{g_0,L_0} g_0\|_{M^1_s} < \delta_3 \ \text{or} \ \|h - S^{-1}_{g,L} g\|_{M^1_s} < \delta_3, \end{aligned} \tag{4.6}$$

implies that, for each $1 \le p \le \infty$ and $0 \le |t| \le s$,

$$\|f - S_{g,h,L} f\|_{M^p_t} \le \varepsilon \|f\|_{M^p_t} \qquad \text{for all } f \in M^p_t(\mathbb{R}^d). \tag{4.7}$$

(iii) *If $g_0 \in \mathscr{S}(\mathbb{R}^d)$, then the assumption*

$$\begin{aligned} &g \to g_0 \ \text{in } \mathscr{S}(\mathbb{R}^d), \\ &L \to L_0 \ \text{in } GL(\mathbb{R}^{2d}), \ \text{and} \\ &h \to S^{-1}_{g_0,L_0} g_0 \ \text{in } \mathscr{S}(\mathbb{R}^d) \end{aligned} \tag{4.8}$$

implies that

$$S_{g,h,L}f \to f \ \text{in}\ \mathscr{S}(\mathbb{R}^d) \qquad \text{for all}\ f \in \mathscr{S}(\mathbb{R}^d). \tag{4.9}$$

Proof. (i), (ii) These statements with the first respective condition on h are an explicit version of Thm. 3.6(i). The alternative condition on h is justified by Thm. 1.1.

(iii) Statement (iii) follows from statement (ii) with $p = 1$ and $t = s$, in view of the relation (4.1) between the modulation spaces and the Schwartz space, as pointed out in the proof of Corollary 1.2(ii). □

Remark 4.2. (i) In Thm. 4.1(i) and (ii) the second respective condition on h, which is that $h - S_{g,L}^{-1}g$ is small enough, does not require explicit knowledge of the actual dual atom $\widetilde{g}_0$ at (g_0, L_0) as long as the dual atom for the approximate case (g, L) is known.

(ii) For Gabor analysis in connection with L^p-spaces, see [23, 26, 27, 39]. We point out that the stability in Thm. 4.1(i) holds in the operator norm on any $L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, provided the Gabor atom is contained in Feichtinger's algebra $M^1(\mathbb{R}^d)$. We note that the frame-type operator $S_{g,h,L}$ in (4.5) is originally defined as a series,

$$S_{g,h,L}f = \sum_{k \in \mathbb{Z}^{2d}} \langle f, \pi(Lk)h\rangle \pi(Lk)g, \qquad f \in L^p(\mathbb{R}^d), \tag{4.10}$$

which converges in the L^p-sense for $p = 2$ but in general not for $p \neq 2$, see the comments in [21, p. 143]. For $p \neq 2$ the series is still well defined with unconditional w^*-convergence on the modulation space $M^\infty(\mathbb{R}^d)$ (alias $S_0'(\mathbb{R}^d)$) [21, Corollary 3.3.3(i)(c)]. However, in our approach the operator $S_{g,h,L}$ is written in the form of the Janssen representation (3.7), which for $g, h \in M^1(\mathbb{R}^d)$ is an absolutely convergent series in the operator norm on $L^p(\mathbb{R}^d)$ for $1 \le p \le \infty$.

5. Further remarks and examples

We exemplify Thm. 1.1 for the standard case of the time-frequency lattices $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ in dimension $d = 1$.

Example 5.1. If $g_0 \in M^1(\mathbb{R})$ generates a Gabor frame for the time-frequency lattice $\Lambda_0 = a_0\mathbb{Z} \times b_0\mathbb{Z}$, $a_0, b_0 > 0$, then there exist $\delta_1, \delta_2, \delta_3 > 0$ such that all $g \in M^1(\mathbb{R})$ with $\|g - g_0\|_{M^1(\mathbb{R})} < \delta_1$ generate a Gabor frame for all lattice constants $a, b > 0$ with $|a - a_0| < \delta_2$ and $|b - b_0| < \delta_3$. Moreover, the canonical dual atom $\widetilde{g}$ for each of these Gabor frames depends continuously in the $M^1(\mathbb{R})$-norm on $g \in M^1(\mathbb{R})$, a and b.

Proof. We have $\Lambda_0 = L_0\mathbb{Z}^2$ with $L_0 = \left(\begin{smallmatrix} a_0 & 0 \\ 0 & b_0 \end{smallmatrix}\right)$. For $L = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, $a, b > 0$, the condition $|a - a_0| \to 0$ and $|b - b_0| \to 0$ yields $L \to L_0$ in $GL(\mathbb{R}^2)$, so the statement follows from Thm. 1.1. □

Remark 5.2. (i) The minimal condition on the Gabor atom such that Thm. 1.1 applies is the unweighted case $s = 0$. In particular, a Gabor frame is stable under a perturbation of the lattice parameters provided that the Gabor atom g belongs to Feichtinger's algebra $M^1(\mathbb{R}^d)$. Various characterizations of $M^1(\mathbb{R}^d)$ are given in [21], see also [25, Sec. 12.1]. Useful sufficient conditions are collected in [21, Thm. 3.2.17], and the following is proved in [35]. For $d = 1$ the Sobolev space with exponent $p = 1$ and two derivatives is continuously embedded into $M^1(\mathbb{R})$. In other words, there is a constant $C > 0$ such that if $f, f', f'' \in L^1(\mathbb{R})$ then $f \in M^1(\mathbb{R})$ with

$$\|f\|_{M^1} \le C\big(\|f\|_{L^1} + \|f'\|_{L^1} + \|f''\|_{L^1}\big). \tag{5.1}$$

(ii) The convergence in $M^1(\mathbb{R}^d)$ implies convergence in $L^1(\mathbb{R}^d)$ as well as the uniform convergence, hence also convergence in $L^2(\mathbb{R}^d)$. However, in contrast to convergence in $L^2(\mathbb{R}^d)$ alone, we note that the $M^1(\mathbb{R}^d)$ convergence of Gabor atoms implies the operator norm convergence of the corresponding frame-type operators [21, Corollary 3.3.3(i)(b)].

In the following example we keep the Gabor atom fixed and vary the lattice constants only.

Example 5.3. For general Gabor atoms $g \in L^2(\mathbb{R})$ the set

$$\begin{aligned} F_g = \big\{(a,b) \in \mathbb{R}_+ \times \mathbb{R}_+ \colon (g,a,b) \text{ generates a} \\ \text{Gabor frame } \{\pi(ak_1, bk_2)g\}_{(k_1,k_2)\in\mathbb{Z}^2}\big\} \end{aligned} \tag{5.2}$$

may have a strange shape. For example, the characteristic function of an interval yields for F_g the peculiar pattern of Janssen's tie [33], which is not an open set. In contrast, Thm. 1.1 implies that for Gabor atoms g in $M^1(\mathbb{R})$ the set F_g is open in $\mathbb{R}_+ \times \mathbb{R}_+$. We verify this statement for three examples. The set F_g has been completely determined for the Gaussian function $g_1(x) = e^{-\pi x^2}$ and the hyperbolic secant function $g_2(x) = (\cosh \pi x)^{-1}$, which are in $\mathscr{S}(\mathbb{R})$, and for the two-sided exponential function $g_3(x) = e^{-|x|}$, which belongs to $M^1(\mathbb{R})$. In fact, g_i for each $i = 1, 2, 3$ generates a frame for any $a, b > 0$ with $ab < 1$ [25, Thm. 7.5.3], [34], [32, Sec. 5], that is,

$$F_{g_1} = F_{g_2} = F_{g_3} = \{(a,b) \in \mathbb{R}_+ \times \mathbb{R}_+ \colon ab < 1\}, \tag{5.3}$$

which is indeed an open set.

Our second main result, Thm. 1.1(ii), implies that the mapping from the lattice constants (a, b) to the canonical dual atom $\widetilde{g}$ is continuous from F_g into $M^1(\mathbb{R})$ for $g = g_3 \in M^1(\mathbb{R})$; and for $g = g_1, g_2 \in \mathscr{S}(\mathbb{R})$ this mapping is even continuous from F_g into $\mathscr{S}(\mathbb{R})$ by Corollary 1.2. Gabor frames generated by the Gaussian function $g = g_1$ are a classical object in time-frequency analysis. They have also been investigated in signal theory and as coherent states in quantum mechanics. The continuous dependence of the dual atom is a new result even for this case.

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